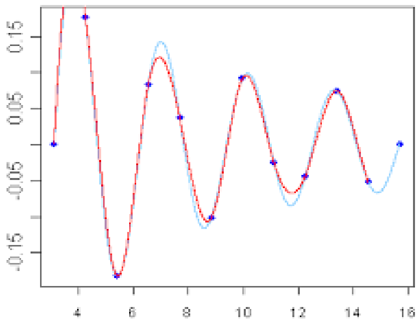
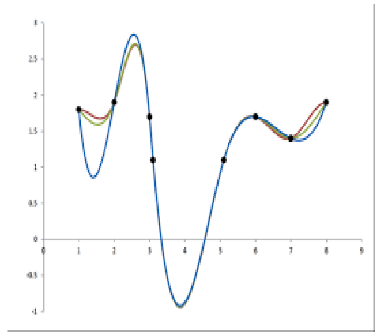
# Lecture 12: Interpolation



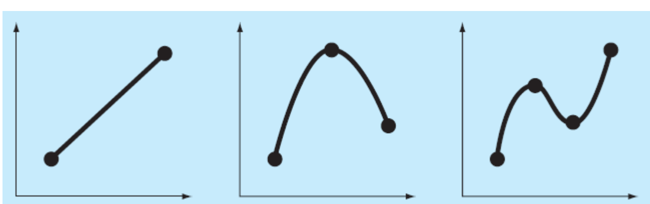





# Polynomial Interpolation

- Polynomial interpolation consists of determining the unique *n*th-order polynomial that fits *n* + 1 data points.
- This polynomial then provides a formula to compute intermediate values.

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$



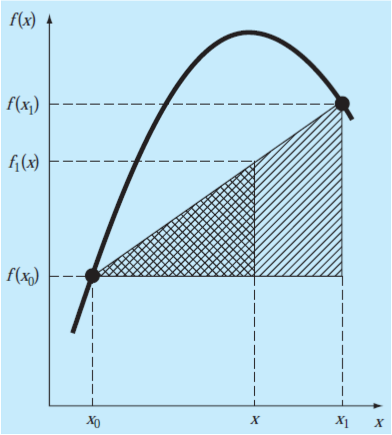
# Our Focus!

- Newton polynomials
- Lagrange polynomials



# Newton's Divided-difference Interpolating Polynomials

- **Linear Interpolation**




$$\frac{f_1(x) - f(x_0)}{x - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

Finite-divided-difference approximation of the first derivative

# Example 1

Problem Statement. Estimate the natural logarithm of 2 using linear interpolation. First, perform the computation by interpolating between ln 1 = 0 and ln 6 = 1.791759. Then, repeat the procedure, but use a smaller interval from ln 1 to ln 4 (1.386294). Note that the true value of ln 2 is 0.6931472.

$x_0 = 1$ to $x_1 = 6$

$$f_1(2) = 0 + \frac{1.791759 - 0}{6 - 1}(2 - 1) = 0.3583519$$

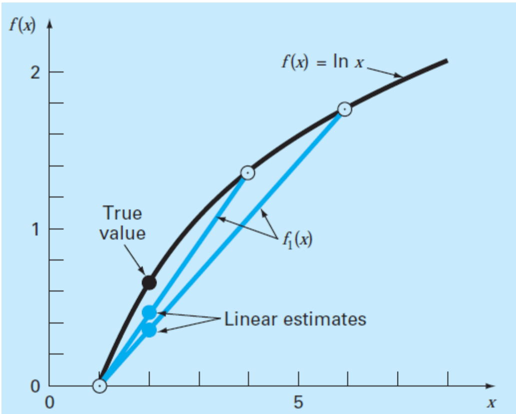
$\varepsilon_t = 48.3\%$.

$x_0 = 1$ to $x_1 = 4$

$$f_1(2) = 0 + \frac{1.386294 - 0}{4 - 1}(2 - 1) = 0.4620981 \qquad \varepsilon_t = 33.3\%.$$



# Example 1



# Quadratic Interpolation

- If three data points are available

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$



# Example 2

Problem Statement. Estimate the natural logarithm of 2 using linear interpolation. First, perform the computation by interpolating between ln 1 = 0 and ln 6 = 1.791759. Then, repeat the procedure, but use a smaller interval from ln 1 to ln 4 (1.386294). Note that the true value of ln 2 is 0.6931472.

Problem Statement. Fit a second-order polynomial to the three points used in Example 18.1:

$x_0 = 1 \qquad f(x_0) = 0$

$x_1 = 4 \qquad f(x_1) = 1.386294$

$x_2 = 6 \qquad f(x_2) = 1.791759$

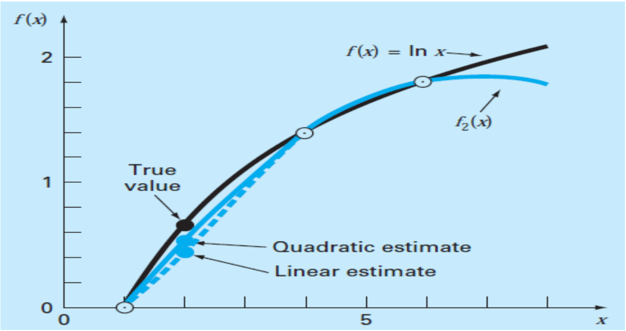
$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

Use the polynomial to evaluate ln 2.

$$b_0 = 0 \qquad b_1 = \frac{1.386294 - 0}{4 - 1} = 0.4620981$$

$$b_2 = \frac{\dfrac{1.791759 - 1.386294}{6 - 4} - 0.4620981}{6 - 1} = -0.0518731$$

# Example 2




$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$f_2(x) = 0 + 0.4620981(x - 1) - 0.0518731(x - 1)(x - 4)$$

at $x = 2$ $\quad f_2(2) = 0.5658444 \quad \varepsilon_t = 18.4\%.$



# General Form of Newton's Interpolating Polynomials

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

⇩

$$f_n(x) = b_0 + b_1(x - x_0) + \cdots + b_n(x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

$$\begin{aligned} b_0 &= f(x_0) \\ b_1 &= f[x_1, x_0] \\ b_2 &= f[x_2, x_1, x_0] \\ &\vdots \\ b_n &= f[x_n, x_{n-1}, \ldots, x_1, x_0] \end{aligned}$$

# Newton's divided-difference interpolating polynomial

$$f_n(x) = b_0 + b_1(x - x_0) + \cdots + b_n(x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

$$f[x_i, x_j] = \frac{f(x_i) - f(x_j)}{x_i - x_j}$$

$$f[x_i, x_j, x_k] = \frac{f[x_i, x_j] - f[x_j, x_k]}{x_i - x_k}$$

$$f[x_n, x_{n-1}, \ldots, x_1, x_0] = \frac{f[x_n, x_{n-1}, \ldots, x_1] - f[x_{n-1}, x_{n-2}, \ldots, x_0]}{x_n - x_0}$$

$$\begin{aligned} f_n(x) = {} & f(x_0) + (x - x_0)\, f[x_1, x_0] + (x - x_0)(x - x_1)\, f[x_2, x_1, x_0] \\ & + \cdots + (x - x_0)(x - x_1)\cdots(x - x_{n-1})\, f[x_n, x_{n-1}, \ldots, x_0] \end{aligned}$$



# Recursive nature of finite divided differences

| $i$ | $x_i$ | $f(x_i)$ | First | Second | Third |
|---|---|---|---|---|---|
| 0 | $x_0$ | $f(x_0)$ | $f[x_1, x_0]$ | $f[x_2, x_1, x_0]$ | $f[x_3, x_2, x_1, x_0]$ |
| 1 | $x_1$ | $f(x_1)$ | $f[x_2, x_1]$ | $f[x_3, x_2, x_1]$ | |
| 2 | $x_2$ | $f(x_2)$ | $f[x_3, x_2]$ | | |
| 3 | $x_3$ | $f(x_3)$ | | | |

# Example 3

Problem Statement. In Example 18.2, data points at $x_0 = 1$, $x_1 = 4$, and $x_2 = 6$ were used to estimate ln 2 with a parabola. Now, adding a fourth point $[x_3 = 5;\ f(x_3) = 1.609438]$, estimate ln 2 with a third-order Newton's interpolating polynomial.

| | |
|---|---|
| $x_0 = 1$ | $f(x_0) = 0$ |
| $x_1 = 4$ | $f(x_1) = 1.386294$ |
| $x_2 = 6$ | $f(x_2) = 1.791759$ |

$$f_3(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$

First divided differences:

$$f[x_1, x_0] = \frac{1.386294 - 0}{4 - 1} = 0.4620981$$

$$f[x_2, x_1] = \frac{1.791759 - 1.386294}{6 - 4} = 0.2027326$$

$$f[x_3, x_2] = \frac{1.609438 - 1.791759}{5 - 6} = 0.1823216$$



# Example 3

Problem Statement. In Example 18.2, data points at $x_0 = 1$, $x_1 = 4$, and $x_2 = 6$ were used to estimate ln 2 with a parabola. Now, adding a fourth point $[x_3 = 5;\ f(x_3) = 1.609438]$, estimate ln 2 with a third-order Newton's interpolating polynomial.

| | |
|---|---|
| $x_0 = 1$ | $f(x_0) = 0$ |
| $x_1 = 4$ | $f(x_1) = 1.386294$ |
| $x_2 = 6$ | $f(x_2) = 1.791759$ |

$$f_3(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$

Second divided differences:

$$f[x_2, x_1, x_0] = \frac{0.2027326 - 0.4620981}{6 - 1} = -0.05187311$$

$$f[x_3, x_2, x_1] = \frac{0.1823216 - 0.2027326}{5 - 4} = -0.02041100$$

# Example 3

Problem Statement. In Example 18.2, data points at $x_0 = 1$, $x_1 = 4$, and $x_2 = 6$ were used to estimate ln 2 with a parabola. Now, adding a fourth point $[x_3 = 5;\ f(x_3) = 1.609438]$, estimate ln 2 with a third-order Newton's interpolating polynomial.

| | |
|---|---|
| $x_0 = 1$ | $f(x_0) = 0$ |
| $x_1 = 4$ | $f(x_1) = 1.386294$ |
| $x_2 = 6$ | $f(x_2) = 1.791759$ |

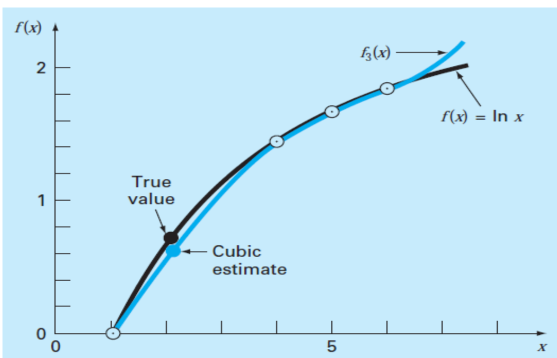
$$f_3(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$

$$f[x_3, x_2, x_1, x_0] = \frac{-0.02041100 - (-0.05187311)}{5 - 1} = 0.007865529$$

**Third divided differences**



# Example 3




$$f_3(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$

$$f_3(x) = 0 + 0.4620981(x - 1) - 0.05187311(x - 1)(x - 4) + 0.007865529(x - 1)(x - 4)(x - 6)$$

$f_3(2) = 0.6287686$ $\varepsilon_t = 9.3\%.$

# Errors of Newton's Interpolating Polynomials

- If the true underlying function is an *n*th-order polynomial, the *n*th-order interpolating polynomial based on $n + 1$ data points will yield exact results.
- The truncation error for the Taylor series:

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x_{i+1} - x_i)^{n+1}$$

- For an *n*th-order interpolating polynomial, an analogous relationship for the error:

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)$$

**The function must be known and differentiable.**



# Errors of Newton's Interpolating Polynomials

- Alternative formulation:

$$R_n = f[x, x_n, x_{n-1}, \ldots, x_0](x - x_0)(x - x_1)\cdots(x - x_n)$$

where $f[x, x_n, x_{n-1}, \ldots, x_0]$ is the $(n+1)$th finite divided difference.

- This contains the unknown *f*(*x*), it cannot be solved for the error.
- However, if an additional data point *f* ($x_n$+1) is available, it can be used to estimate the error:

$$R_n \cong f[x_{n+1}, x_n, x_{n-1}, \ldots, x_0](x - x_0)(x - x_1)\cdots(x - x_n)$$

# Example 4

Problem Statement. Use Eq. (18.18) to estimate the error for the second-order polynomial interpolation of Example 18.2. Use the additional data point $f(x_3) = f(5) = 1.609438$ to obtain your results.

$$R_2 = f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)$$

$$R_2 = 0.007865529(x - 1)(x - 4)(x - 6)$$

at $x = 2$ $\quad R_2 = 0.007865529(2 - 1)(2 - 4)(2 - 6) = 0.0629242$

True error: $0.6931472 - 0.5658444 = 0.1273028.$



# Errors of Newton's Interpolating Polynomials

$$R_n = f_{n+1}(x) - f_n(x)$$ ← a future prediction minus a present one.

$$f_{n+1}(x) = f_n(x) + R_n$$

- This error estimate could be less than the true error.
- This would represent a highly unattractive quality if the error estimate were being employed as a stopping criterion.

# Lagrange Interpolating Polynomials

- A reformulation of the Newton polynomial that avoids the computation of divided differences.

$$f_n(x) = \sum_{i=0}^{n} L_i(x)\, f(x_i)$$

$$L_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

- Linear version: ($n$ = 1)

$$f_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

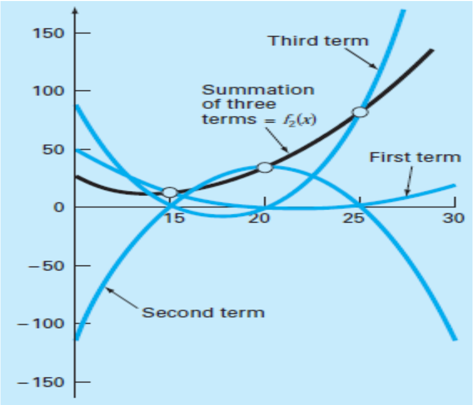
- 2nd order: ($n$ = 2)

$$f_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$



# Rationale behind Lagrange polynomial

$L_i(x)$ will be 1 at $x = x_i$ and 0 at all other sample points



# Derivation from Newton's Interpolating Polynomial

- First-order case: $f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$

⇩

$$f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

⇩

$$f[x_1, x_0] = \frac{f(x_1)}{x_1 - x_0} + \frac{f(x_0)}{x_0 - x_1} \qquad \textit{symmetric form}$$

$$f_1(x) = f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1) + \frac{x - x_0}{x_0 - x_1} f(x_0)$$

$$f_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$



# Example 5

Problem Statement. Use a Lagrange interpolating polynomial of the first and second order to evaluate ln 2 on the basis of the data given in Example 18.2:

$x_0 = 1 \qquad f(x_0) = 0$

$x_1 = 4 \qquad f(x_1) = 1.386294$

$x_2 = 6 \qquad f(x_2) = 1.791760$

$$f_1(2) = \frac{2-4}{1-4}0 + \frac{2-1}{4-1}1.386294 = 0.4620981$$

$$f_2(2) = \frac{(2-4)(2-6)}{(1-4)(1-6)}0 + \frac{(2-1)(2-6)}{(4-1)(4-6)}1.386294 + \frac{(2-1)(2-4)}{(6-1)(6-4)}1.791760 = 0.5658444$$

# Estimated Error

$$R_n = f[x, x_n, x_{n-1}, \ldots, x_0] \prod_{i=0}^{n} (x - x_i)$$



# Comparison between Newton and Lagrange Polynomials

- If the order of the polynomial is unknown, the Newton method has advantages → for exploratory computations, Newton's method is often preferable.
- If only one interpolation is to be performed, the Lagrange and Newton formulations require comparable computational effort.
- However, the Lagrange version is somewhat easier to program.
- The Lagrange form is often used when the order of the polynomial is known a priori.

# Coefficients of an Interpolating Polynomial

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

$$f(x) = a_0 + a_1 x + a_2 x^2$$

$$f(x_0) = a_0 + a_1 x_0 + a_2 x_0^2$$
$$f(x_1) = a_0 + a_1 x_1 + a_2 x_1^2$$
$$f(x_2) = a_0 + a_1 x_2 + a_2 x_2^2$$

**Solve by any elimination method for the coefficients**



# Inverse Interpolation

$$f(x) = 1/x,$$

| $x$ | 1 | 2 | 3 | **?** | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 1 | 0.5 | 0.3333 | **0.3** | 0.25 | 0.2 | 0.1667 | 0.1429 |

| $f(x)$ | 0.1429 | 0.1667 | 0.2 | 0.25 | 0.3333 | 0.5 | 1 |
|---|---|---|---|---|---|---|---|
| $x$ | 7 | 6 | 5 | 4 | 3 | 2 | 1 |

three points: (2, 0.5), (3, 0.3333) and (4, 0.25)

$$f_2(x) = 1.08333 - 0.375x + 0.041667x^2$$

$$0.3 = 1.08333 - 0.375x + 0.041667x^2$$

$$x = \frac{0.375 \pm \sqrt{(-0.375)^2 - 4(0.041667)0.78333}}{2(0.041667)} = \begin{matrix} 5.704158 \\ \boxed{3.295842} \end{matrix}$$

# Assignment-12

- Problems 12.6, 12.8, 18.10, 18.11, 18.12.